The Asymmetric Simple Exclusion Process: An Exactly Solvable Model of Particle Transport

Arvind Ayyer, Indian Institute of Science

28th Mid Year Meeting Faculty Hall, Indian Institute of Science

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Multispecies ASEP

Affine Weyl Groups 000 000000

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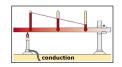
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Nonequilibrium statistical physics











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Introduction		
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Motivation		

Statistical physics

- Systems with many degrees of freedom
- Collective behaviour
- Universality

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Equilibrium statistical physics

- System is in thermodynamic equilibrium.
- Microscopic motion may be present, but macroscopic observables do not change over time.
- The probability distribution depends only on macroscropic observables.
- Example: Gas in a closed box.
- Well-established theory due to Boltzmann and Gibbs (late 1800s early 1900s).

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Introduction		
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Motivation		

Driven systems

- System in contact with more than one reservoir.
- Reservoirs have different values for macroscopic observables.
- Presence of a steady current.
- No fundamental theory.

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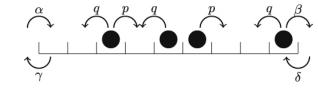
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Asymmetric Simple Exclusion Process (ASEP)

- A one-dimensional lattice of size L.
- Each site is either occupied or empty.



• Take p = 1

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Toy Model		

Totally Asymmetric Simple Exclusion Process (TASEP)

- Special case of ASEP with $q = \gamma = \delta = 0$.
- Long-term behaviour given by the steady state
- Solved exactly by Derrida, Evans, Hakim & Pasquier (1993)

Asymmetric Simple Exclusion Process

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Toy Model		

Matrix Ansatz

• Suppose you find matrices X_0, X_1 and vectors $\langle W |, |V \rangle$ s.t.

$$X_1X_0 = X_1 + X_0$$
 $X_1|V\rangle = rac{1}{eta}|V
angle$ $\langle W|X_0 = rac{1}{lpha}\langle W|,$

•
$$\underline{\tau} = (\tau_1, \dots, \tau_L)$$
 be a configuration; $\tau_i \in \{0, 1\}$.

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Matrix Ansatz

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angle$ $\langle W|X_0 = rac{1}{lpha}\langle W|,$

- $\underline{\tau} = (\tau_1, \dots, \tau_L)$ be a configuration; $\tau_i \in \{0, 1\}$.
- The steady state probability of $\underline{\tau}$ is given by

$$P(\underline{\tau}) = \frac{1}{Z_L} \langle W | X_{\tau_1} \dots X_{\tau_L} | V \rangle.$$

• Z_L is the normalization factor

$$Z_L = \langle W | (X_0 + X_1)^L | V \rangle.$$

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The Representation

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Toy Model		

Applications: Explicit formulas

Normalisation factor

$$Z_{L} = \sum_{i=1}^{L} \frac{i (2L - 1 - i)!}{L! (L - i)!} \frac{\beta^{-i-1} - \alpha^{-i-1}}{\beta^{-1} - \alpha^{-1}}.$$

$$J = \frac{Z_{L-1}}{Z_L}$$

Density

$$\rho_{i} = \sum_{j=0}^{L-i-1} \frac{(2j)!}{j! \ (j+1)!} \frac{Z_{L-j-1}}{Z_{L}} + \frac{Z_{i-1}}{Z_{L}} \sum_{j=2}^{L-i+1} \frac{(j-1) \ (2L-2i-j)!}{(L-i)! \ (L-i-j+1)!} \beta^{-j}$$

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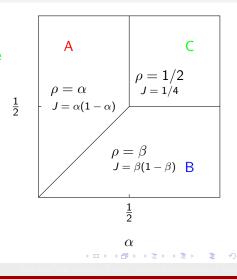
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Phase Diagram

A is the low density phase B is the high density phase C is the maximal current phase

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More Applications: Out-of-equilibrium observables

- Diffusion constant (Derrida, Evans & Mallick, '95)
- Large deviation functional of the density (Derrida, Lebowitz & Speer, '03)
- Spectrum of the generator (de Gier & Essler, '05)
- Large deviation function of the current (Lazarescu & Mallick, '11)

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Multispecies ASEP

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Multispecies ASEP
 Semipermeable

Multispecies





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Semipermeable ASEP

- Particles of type +, as well as vacancies.
- Number *n*₀ of vacancies conserved.
- Bulk rules (action of an electric field)

$$+ 0 \frac{1}{\sqrt{q}} 0 + 0 - \frac{1}{\sqrt{q}} - 0 + \frac{1}{\sqrt{q}} - 0 + \frac{1}{\sqrt{q}} - 1 + \frac{1}{\sqrt{q}} - 1 + \frac{1}{\sqrt{q}} - 1 + \frac{1}{\sqrt{q}} +$$

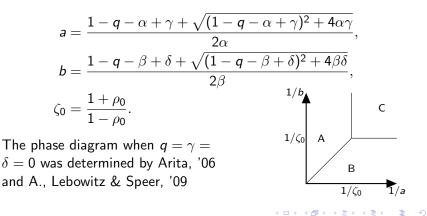
• Left/Right boundary

$$-\frac{\alpha}{\gamma}+, +\frac{\beta}{\delta}$$

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The Phase Diagram

Take $L \to \infty$ such that $n_0/L \to \rho_0$. (Uchiyama, '08)



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The mASEP

- Introduced by Cantini, Garbali, de Gier & Wheeler, '16.
- One-dimensional lattice of size L
- r species of charges, denoted j and $\overline{j} \equiv -j$, and 0's.
- Total number of charges of species j is n_j .
- Number n₀ of vacancies conserved.
- Bulk rules (action of an electric field)

$$j k \stackrel{1}{\rightleftharpoons}_{q} k j \quad \text{if } j > k$$

Left/Right boundary

$$\overline{j} \stackrel{\alpha}{\underset{\gamma}{\rightleftharpoons}} j, \qquad j \stackrel{\beta}{\underset{\delta}{\longleftarrow}} \overline{j}$$

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Thermodynamic Limit

• Take $L \to \infty$ and $n_j \to \infty$ for each j such that $n_j/L \to \theta_j > 0$.

•
$$\Theta_k = (\theta_k + \cdots + \theta_r)/2$$

•
$$\phi_k = \Theta_k/(1-\Theta_k)$$
 for $1 \le k \le r$

• Let
$$f(x) = 1/(1+x)$$
.

Theorem (A and D. Roy, arXiv:1611.01943)

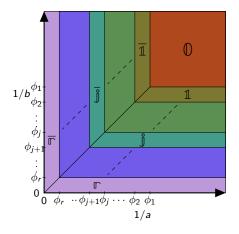
The phase diagram of the mASEP with r species of charges is as follows.

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Phase diagram



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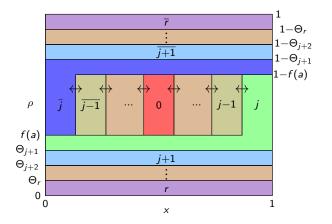
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Affine Weyl Groups 000 000000

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Multispecies

Snapshot on the $j - \overline{j}$ boundary



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 - Multispecies TASEP

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Setting				

Terminology

- W a finite Weyl group
- \hat{W} the corresponding affine Weyl group, acting on V
- H the corresponding affine Coxeter arrangement
- Connected components of $V \setminus H$ are the alcoves
- T. Lam proposed a random walk on the alcoves:
 - Start at the fundamental alcove
 - At each step, we cross a uniformly random adjacent hyperplane,
 - Subject to the condition that we never cross a hyperplane twice.

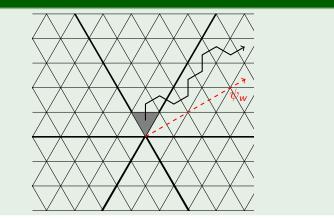
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Setting

Multispecies ASEP

The \tilde{A}_2 arrangement

Example



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Setting			

Limiting Direction

- Lam (2014) conjectured a formula for the limiting direction ψ_w when $W = A_n$.
- The formula involved a finite state Markov chain on S_n .
- Unknown to him, this was studied earlier ...

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Limiting Direction

- Lam (2014) conjectured a formula for the limiting direction ψ_w when $W = A_n$.
- The formula involved a finite state Markov chain on S_n .
- Unknown to him, this was studied earlier ...
- A multispecies TASEP with periodic boundary conditions.

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Multispecies TASEP

Multispecies TASEP on the ring

- n+1 species of particles on L sites $(L+1 \equiv 1)$.
- Each site occupied by one particle.
- Vacancies are particles of type n + 1.
- Number of each species is conserved
- The following transition takes place

 $\alpha\beta\to\beta\alpha$ with rate 1 if $\alpha<\beta$

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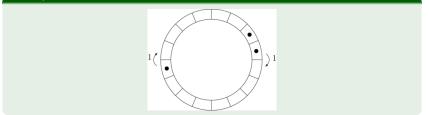
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Special case: n = 1

Example



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Steady State

- Uniform when n = 1, but not in general
- Ferrari and Martin (2007) gave a complete solution for arbitrary *n* and *L*
- The solution involves multiclass M/M/1 queues

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Two-point correlations

Multispecies TASEP

- Let L = n, and one particle of each species $\{1, \ldots, n\}$
- Lam's conjecture involves the calculation of $E_{i,i}$, the steady-state probability of particle *i* at site 1 and particle *j* at site 2

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Main result

Theorem (A & S. Linusson, Trans. of the AMS, 2017)

For any $1 \le i \le j \le n$, we have

$$E_{j,i} = \frac{j-i}{n\binom{n}{2}},$$

$$E_{i,j} = \begin{cases} \frac{1}{n^2} + \frac{i(n-i)}{n^2(n-1)}, & \text{if } i = j-1, \\ \\ \frac{1}{n^2}, & \text{if } i < j-1. \end{cases}$$

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Example

Values of $n\binom{n}{2} \times E_{w_1,w_2}$ for n = 5

$w_1 \setminus w_2$	1	2	3	4	5
1	0	4	2	2	2
2	1	0	5	2	2
3	2	1	0	5	2
4	3	2	1	0	4
5	4	3	2	1	0

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Introduction

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Affine Weyl Groups