The Asymmetric Simple Exclusion Process: An Exactly Solvable Model of Particle Transport

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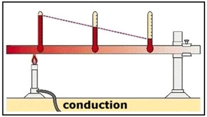
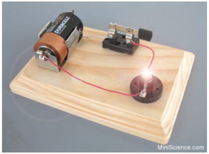
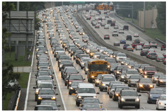
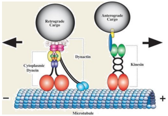
Nonequilibrium statistical physics









Statistical physics

- Systems with many degrees of freedom
- Collective behaviour
- Universality

Equilibrium statistical physics

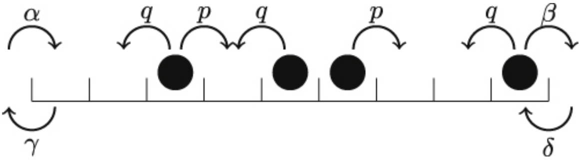
- System is in thermodynamic equilibrium.
- Microscopic motion may be present, but macroscopic observables do not change over time.
- The probability distribution depends only on macroscropic observables.
- Example: Gas in a closed box.
- Well-established theory due to Boltzmann and Gibbs (late 1800s - early 1900s).

Driven systems

- System in contact with more than one reservoir.
- Reservoirs have different values for macroscopic observables.
- Presence of a steady current.
- No fundamental theory.

Asymmetric Simple Exclusion Process (ASEP)

- A one-dimensional lattice of size L.
- Each site is either occupied or empty.



- Take $p = 1$

Totally Asymmetric Simple Exclusion Process (TASEP)

- Special case of ASEP with $q = \gamma = \delta = 0$.
- Long-term behaviour given by the steady state
- Solved exactly by Derrida, Evans, Hakim & Pasquier (1993)

Matrix Ansatz

- Suppose you find matrices X_0, X_1 and vectors $\langle W|, |V\rangle$ s.t.

$$X_1 X_0 = X_1 + X_0 \qquad X_1|V\rangle = \frac{1}{\beta}|V\rangle \qquad \langle W|X_0 = \frac{1}{\alpha}\langle W|,$$

- $\underline{\tau} = (\tau_1, \dots, \tau_L)$ be a configuration; $\tau_i \in \{0, 1\}$.

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- $\underline{\tau} = (\tau_1, \dots, \tau_L)$ be a configuration; $\tau_i \in \{0,1\}$.
- The steady state probability of $\underline{\tau}$ is given by

$$P(\underline{\tau}) = \frac{1}{Z_L}\langle W|X_{\tau_1} \dots X_{\tau_L}|V\rangle.$$

- Z_L is the normalization factor

$$Z_L = \langle W|(X_0 + X_1)^L|V\rangle.$$

The Representation

$$X_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & \cdot & \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \end{pmatrix}, \qquad X_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot \\ 1 & 1 & 0 & 0 & & \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ \cdot & & & \cdot & \cdot & \\ \cdot & & & & \cdot & \cdot \end{pmatrix}.$$

$$\langle W_\alpha | = \left(1, \left(\frac{1-\alpha}{\alpha} \right), \left(\frac{1-\alpha}{\alpha} \right)^2, \dots \right), | V_\beta \rangle = \langle W_\beta |^{\mathrm{T}}$$

Applications: Explicit formulas

- Normalisation factor

$$Z_L = \sum_{i=1}^{L} \frac{i\,(2L-1-i)!}{L!\,(L-i)!} \frac{\beta^{-i-1} - \alpha^{-i-1}}{\beta^{-1} - \alpha^{-1}}.$$

- Current

$$J = \frac{Z_{L-1}}{Z_L}$$

- Density

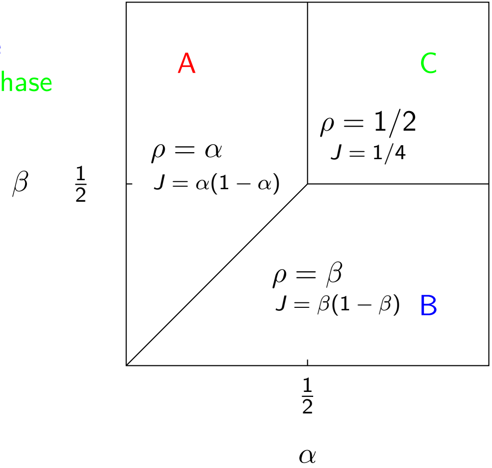
$$\rho_i = \sum_{j=0}^{L-i-1} \frac{(2j)!}{j!\,(j+1)!} \frac{Z_{L-j-1}}{Z_L} + \frac{Z_{i-1}}{Z_L} \sum_{j=2}^{L-i+1} \frac{(j-1)\,(2L-2i-j)!}{(L-i)!\,(L-i-j+1)!} \beta^{-j}$$

Phase Diagram

A is the low density phase

B is the high density phase

C is the maximal current phase



More Applications: Out-of-equilibrium observables

- Diffusion constant (Derrida, Evans & Mallick, '95)
- Large deviation functional of the density (Derrida, Lebowitz & Speer, '03)
- Spectrum of the generator (de Gier & Essler, '05)
- Large deviation function of the current (Lazarescu & Mallick, '11)

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Semipermeable ASEP

- Particles of type $+, -$ as well as vacancies.
- Number n_0 of vacancies conserved.
- Bulk rules (action of an electric field)

$$+0 \underset{q}{\overset{1}{\rightleftharpoons}} 0+$$

$$0- \underset{q}{\overset{1}{\rightleftharpoons}} -0$$

$$+- \underset{q}{\overset{1}{\rightleftharpoons}} -+$$

- Left/Right boundary

$$- \underset{\gamma}{\overset{\alpha}{\rightleftharpoons}} +, \qquad + \underset{\delta}{\overset{\beta}{\rightleftharpoons}} -$$

The Phase Diagram

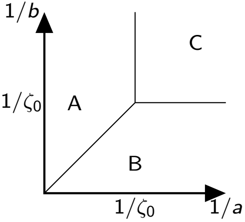
Take $L \to \infty$ such that $n_0/L \to \rho_0$. (Uchiyama, '08)

$$a = \frac{1 - q - \alpha + \gamma + \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha},$$

$$b = \frac{1 - q - \beta + \delta + \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta},$$

$$\zeta_0 = \frac{1 + \rho_0}{1 - \rho_0}.$$

The phase diagram when $q = \gamma = \delta = 0$ was determined by Arita, '06 and A., Lebowitz & Speer, '09



The mASEP

- Introduced by Cantini, Garbali, de Gier & Wheeler, '16.
- One-dimensional lattice of size L
- r species of charges, denoted j and $\bar{j} \equiv -j$, and 0's.
- Total number of charges of species j is n_j.
- Number n_0 of vacancies conserved.
- Bulk rules (action of an electric field)

$$j\,k \underset{q}{\overset{1}{\rightleftharpoons}} k\,j \quad \text{if } j > k$$

- Left/Right boundary

$$\bar{j} \underset{\gamma}{\overset{\alpha}{\rightleftharpoons}} j, \qquad j \underset{\delta}{\overset{\beta}{\rightleftharpoons}} \bar{j}$$

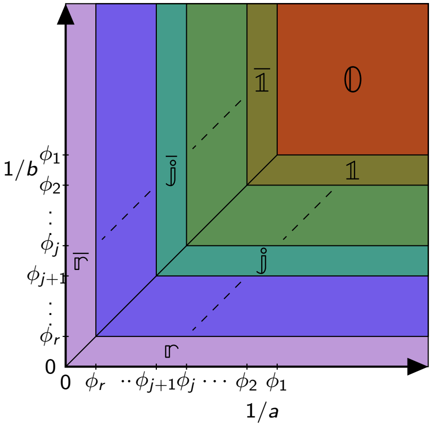
Thermodynamic Limit

- Take $L \to \infty$ and $n_j \to \infty$ for each j such that $n_j/L \to \theta_j > 0$.
- $\Theta_k = (\theta_k + \cdots + \theta_r)/2$
- $\phi_k = \Theta_k/(1 - \Theta_k)$ for $1 \leq k \leq r$
- Let $f(x) = 1/(1 + x)$.

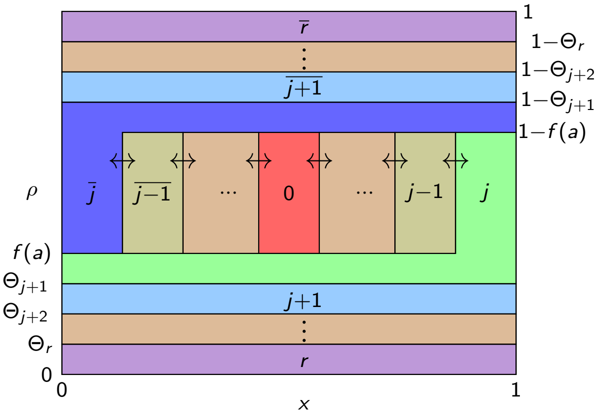
Theorem (A and D. Roy, arXiv:1611.01943)

The phase diagram of the mASEP with r species of charges is as follows.

Phase diagram



Snapshot on the $\mathbb{j} - \bar{\mathbb{j}}$ boundary



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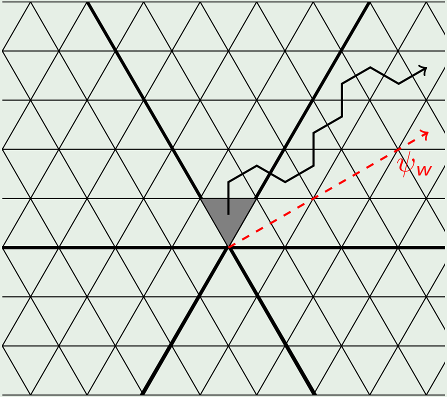
Terminology

- W - a finite Weyl group
- $\hat{W}$ - the corresponding affine Weyl group, acting on V
- H - the corresponding affine Coxeter arrangement
- Connected components of $V \setminus H$ are the alcoves
- T. Lam proposed a random walk on the alcoves:
    1. Start at the fundamental alcove
    2. At each step, we cross a uniformly random adjacent hyperplane,
    3. subject to the condition that we never cross a hyperplane twice.

The $\tilde{A}_2$ arrangement

Example



Limiting Direction

- Lam (2014) conjectured a formula for the limiting direction ψ_w when $W = A_n$.
- The formula involved a finite state Markov chain on S_n.
- Unknown to him, this was studied earlier ...

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- Lam (2014) conjectured a formula for the limiting direction ψ_w when $W = A_n$.
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- Unknown to him, this was studied earlier ...
- A multispecies TASEP with periodic boundary conditions.

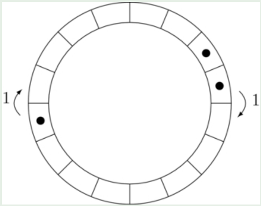
Multispecies TASEP on the ring

- $n+1$ species of particles on L sites $(L+1 \equiv 1)$.
- Each site occupied by one particle.
- Vacancies are particles of type $n+1$.
- Number of each species is conserved
- The following transition takes place

$$\alpha\beta \to \beta\alpha \text{ with rate 1 if } \alpha < \beta$$

Special case: $n = 1$

Example



Steady State

- Uniform when $n = 1$, but not in general
- Ferrari and Martin (2007) gave a complete solution for arbitrary n and L
- The solution involves multiclass $M/M/1$ queues

Two-point correlations

- Let $L = n$, and one particle of each species $\{1, \ldots, n\}$
- Lam's conjecture involves the calculation of $E_{i,j}$, the steady-state probability of particle i at site 1 and particle j at site 2

Main result

Theorem (A & S. Linusson, *Trans. of the AMS*, 2017)

For any $1 \leq i \leq j \leq n$, *we have*

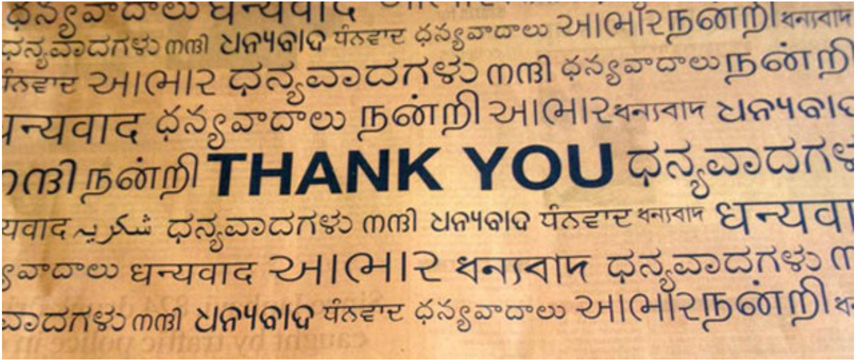
$$E_{j,i} = \frac{j-i}{n\binom{n}{2}},$$

$$E_{i,j} = \begin{cases} \dfrac{1}{n^2} + \dfrac{i(n-i)}{n^2(n-1)}, & \text{if } i = j-1, \\[2ex] \dfrac{1}{n^2}, & \text{if } i < j-1. \end{cases}$$

Example

Values of $n\binom{n}{2} \times E_{w_1,w_2}$ for $n = 5$

$w_1 \backslash w_2$	1	2	3	4	5
1	0	4	2	2	2
2	1	0	5	2	2
3	2	1	0	5	2
4	3	2	1	0	4
5	4	3	2	1	0


THANK YOU